Universalidade da Beira Interior

MSc in Aerospace Engineering

Astrodynamics

Kalman Filter in Attitude Determination

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1. Introduction

The Kalman filter has been used in numerous fields of study such as avionics, navigation systems, control systems and trajectory optimization as a result of its powerful optimal estimation algorithm which is used to estimate states of a system from direct or indirect measurements with a high level of uncertainty. The filter is named after Rudolf E. Kálmán, the first person to mathematically formalize the theory.

The motivation of Kalman in 1960\cite{2} to develop the filter emerged when he identified three problems related with the statistical nature of theoretical and practical problems in communication and control, which were:

i) Prediction of random signals

ii) Separation of random signals from random noise

iii) Detection of signals of known form (pulses, sinusoids) in the presence of random noise.

The algorithm uses a physical model and measurements given by sensors in real time, both affected with uncertainties coming from random disturbances, to produce state estimates that are usually more reliable than those based on the physical model or measurements alone. It is capable to merge the prediction obtained by the dynamical model with the actual measurements based on their uncertainties, thus creating an optimal state estimate.

The Kalman filter is widely used in aeronautics and engineering for two main purposes: combining measurements of the same variables but from different sensors, and for combining an inexact forecast of a system’s state with an inexact measurement of the state. The Kalman filter has also applications in statistics and function approximation.
2. Attitude Representation

Before proceeding to the attitude determination, it is important to define first what is the attitude of a given spacecraft. The attitude of a certain vehicle is normally represented by the Euler angles, in other words, the vehicle’s spacial orientation relatively to a reference frame[3, 4].

In Figure 1, a geocentric equatorial frame is presented along with a reference frame. Let us consider that a satellite is at the end of the vector and that any rotation of the body frame relatively to the reference frame will represent a variation in attitude.

![Reference Frame](image)

Figure 1: Reference Frame

The rotation of the satellite may be expressed as the combination of three single rotations shown below[4].

\[
R_z(\psi) = \begin{bmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
R_y(\theta) = \begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]

\[
R_x(\phi) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix}
\]

where \(\psi\), \(\theta\) and \(\phi\) are yaw, pitch and roll angles, respectively. The rotation matrix will be:

\[
R = R_x(\phi)R_y(\theta)R_z(\psi)
\]
In order to determine the equations of a vehicle’s attitude, we introduce the change rate of Euler angles in the body frame, being them:

- Roll rate - \( p = \dot{\phi} \)
- Pitch rate - \( q = \dot{\theta} \)
- Yaw rate - \( r = \dot{\psi} \)

Using the rotation matrix, we conclude that in the reference frame the equations that give the vehicle’s attitude are:

\[
\begin{align*}
\dot{\phi} &= p + (q \sin \phi + r \cos \phi) \tan \phi \\
\dot{\theta} &= q \cos \phi - r \sin \phi \\
\dot{\psi} &= \frac{q \sin \phi + r \cos \phi}{\cos \theta}
\end{align*}
\]

The integration of the equations mentioned above allows the attitude determination over time. Due to the fact that errors accumulate over time during the numerical integration of the system of differential equations, good results are only achieved for small values of \( \theta \). In fact, when considering a pitch angle of \( 90^\circ \) or \(-90^\circ\), the \( \cos \) and \( \tan \) function are not defined, making it impossible to numerically integrate the differential system.

In order to overcome the existence of singularities in these equations, another system of equations is created. This new procedure makes use of quaternions, developed by Hamilton, to represent the attitude of a certain vehicle.

Quaternions are generally represented in the form:

\[
\vec{\eta} = \eta_0 + \eta_1 \vec{i} + \eta_2 \vec{j} + \eta_3 \vec{k}
\]

where \( \eta_0, \eta_1, \eta_2 \) and \( \eta_3 \) are real numbers, and \( i, j \) and \( k \) are the fundamental quaternion units. It can be shown that the relation between Euler angles and quaternions are

\[
\begin{align*}
\eta_0 &= \cos \left( \frac{\phi}{2} \right) \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{\psi}{2} \right) + \sin \left( \frac{\phi}{2} \right) \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{\psi}{2} \right) \\
\eta_1 &= \sin \left( \frac{\phi}{2} \right) \cos \left( \frac{\theta}{2} \right) \cos \left( \frac{\psi}{2} \right) - \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{\psi}{2} \right) \\
\eta_2 &= \cos \left( \frac{\phi}{2} \right) \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\psi}{2} \right) + \sin \left( \frac{\phi}{2} \right) \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\psi}{2} \right) \\
\eta_3 &= \cos \left( \frac{\phi}{2} \right) \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\psi}{2} \right) - \sin \left( \frac{\phi}{2} \right) \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\psi}{2} \right)
\end{align*}
\]

and

\[
\begin{align*}
\phi &= \arctan2 \left( 2 (\eta_0 \eta_1 + \eta_2 \eta_3), 1 - 2 \left( \eta_1^2 + \eta_2^2 \right) \right) \\
\theta &= \arcsin \left( 2 (\eta_0 \eta_2 - \eta_3 \eta_1) \right) \\
\psi &= \arctan2 \left( 2 (\eta_0 \eta_3 + \eta_1 \eta_2), 1 - 2 \left( \eta_2^2 + \eta_3^2 \right) \right)
\end{align*}
\]
3. Mathematical Formulation

3.1. Dynamical System Model

Considering a system which may be represented by the following linear dynamic system model

\[
\dot{x}(t) = Ax(t) + Bu(t) + v(t) \\
y(t) = Cx(t) + w(t)
\]

- **A** - State transition model
  
The matrix \( A \) has the information about how the system evolves over time.

- **x** - State vector
  
The state vector is the minimum number of variables that characterizes a system. In attitude determination, the state vector contains the Euler angles or the quaternions.

\[
x = \begin{bmatrix} \theta \\ \phi \\ \psi \end{bmatrix} \quad \vee \quad x = \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}
\]

- **B** - Control input model
  
The matrix \( B \) has the control model that relates the control variables with the state evolution over time.

- **u** - Control vector
  
The control vector contains the inputs given to the system in order to control it. In the present work, it may represent the propulsive systems or the deflection of control surfaces if the satellite is within the atmosphere.

- **v** - Process noise \( \sim N(0, Q_k) \)
  
The process noise represents all the uncertainties and errors associated with the system’s operation, such as nonlineairities and unexpected perturbations.

- **y** - Measurements vector
  
The measurements vector contains which state variables are being measured while the system is operating.

- **C** - Observation model
  
The \( C \) matrix is responsible for selecting which variables of the state vector are measured.

- **w** - Measurements noise \( \sim N(0, R_k) \)
  
The measurements noise represents all the uncertainties and errors associated with the measurements, such as sensor precision and unexpected perturbations.
3.2. Kalman Filter

The Kalman filter consists of two steps, these being the prediction and update step\[^{[3]}\]. The algorithm is shown in Figure 2.

Kalman made some considerations about the treatment of process and measurement noise. He assumed that both noises could be approximated by a normal distribution with zero mean and a certain covariance matrix. He also considered that there was no correlation between generated noises in different time steps.

As mentioned before, the Kalman filter is a two-step process. In the prediction step, the filter calculates the estimate of the current state vector and the associated uncertainty. When the system receives the measurement of the selected variables, this prediction is updated using a weighted average.

Before going further to the Kalman Filter equations, it is necessary to define the notations that are being used. Therefore, $\hat{x}_k^-$ represents the state estimate before correction and, $\hat{x}_k^+$ the state estimate after correction.

3.2.1 Prediction Step

The equations of the prediction step are shown below. This step estimates the actual state using the physical model and calculates the uncertainty associated with this estimate.

\[
\begin{align*}
\hat{x}_k^- &= A_{k-1} \hat{x}_{k-1}^+ + B_{k-1} u_{k-1} \\
P_k^- &= A_{k-1} P_{k-1}^+ A_{k-1}^T + Q_k
\end{align*}
\]
3.2.2 Update Step

The update step contains the following expressions:

\[ K_k = P_k^{-1}C_k^T(C_kP_k^{-1}C_k^T + R_k)^{-1} \]
\[ \hat{x}_k^+ = \hat{x}_k^- + K_k(y_k - C_k\hat{x}_k^-) \]
\[ P_k^+ = (I - K_kC_k)P_k^- \]

The first expression represents the filter gain and is responsible for creating a weighted average between the prediction and actual measurements. The second and third equations are simply the correction of the prediction and the calculation of the associated uncertainty, respectively. It is noteworthy to analyze how the filter selects what is the percentage of information gathered from both prediction and measurements.

Let us consider the case where the process noise is zero but the measurements are affected by some perturbations. Calculating the limit

\[ \lim_{P_k^- \to 0} K_k = 0 \]

we conclude that the filter gain is zero. Replacing into the second expression of the update step we have that

\[ \hat{x}_k^+ = \hat{x}_k^- + 0(y_k - C_k\hat{x}_k^-) \]
\[ \hat{x}_k^+ = \hat{x}_k^- \]
\[ \hat{x}_k^- = \hat{x}_k^- \]

concluding that the predicted estimate using the dynamic system model is the true value and therefore, no measurements are required.

Considering another case where the measurements noise is zero but the system is affected by some perturbations. Calculating the limit

\[ \lim_{R_k \to 0} K_k = C_k^{-1} \]

we obtain \( C_k^{-1} \) as the filter gain. Substituting into the second expression of the update step we have that

\[ \hat{x}_k^+ = \hat{x}_k^- + C_k^{-1}(y_k - C_k\hat{x}_k^-) \]
\[ \hat{x}_k^- = C_k^{-1}y_k - \hat{x}_k^- \]
\[ \hat{x}_k^+ = C_k^{-1}y_k = y_k \]

concluding that the measurements represent the true value. The rest of the cases represent a weighted average between both prediction and measurements.
4. Conclusion

Concluding, the Kalman filter is an optimal estimation algorithm used to estimate states of a system which is affected by great noise. It is important to refer that it is an optimal estimation method if the system respects the considerations that Kalman did when formulating his filtering theory. Also, this filter is capable of estimating variables that were not measured or measured indirectly, using the physical model.

As a filter, this algorithm tries to remove as much noise as possible from the final state by calculating a filter gain that represents a weighted average between the prediction and measurements. In other words, the algorithm is capable of fusing the predictions and measurements based on their uncertainty.

The Kalman filter has countless applications in the aerospace field since it can produce estimates within an acceptable level of error. It is vastly used to calculate the attitude of a certain vehicle but also to develop control systems that can manage its spatial orientation, since the filter can be implemented in parallel with a controller in order to develop a LQG (Linear Quadratic Gaussian).
References


